AD-A162 875 FLUCTUATIONS NEAR HOMOGENEOUS STATES OF CHEMICAL REACTIONS WITH DIFFUSION..(U) NORTH CAROLINA UNIV AT CHAPEL HILL CENTER FOR STOCHASTIC PROC.. P KOTELENEZ 1/1

UNCLASSIFIED NOV 85 TR-126 AFOSR-TR-85-1179 F/G 7/4 NL

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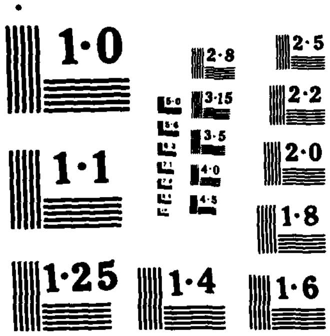
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(2)

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION: UNCLASSIFIED

1b. RESTRICTIVE MARKINGS:

2a. SECURITY CLASSIFICATION AUTHORITY:

2b. DECLASSIFICATION/DOWNGRADING SCHEDULE:

3. DISTRIBUTION/AVAILABILITY OF REPORT: Unlimited. Approved for public release; distribution unlimited.

4. PERFORMING ORGANIZATION REPORT NUMBER(S): Technical Report No. 126

5. MONITORING ORGANIZATION REPORT NUMBER(S): AFOSR-TR- 85-1179

6a. NAME OF PERFORMING ORGANIZATION: Center for Stochastic Processes

6b. OFFICE SYMBOL *(If applicable)*:

7a. NAME OF MONITORING ORGANIZATION: Air Force Office of Scientific Research

6c. ADDRESS *(City, State and ZIP Code)*:
Statistics Dept., Univ. of N. Carolina
Phillips Hall 039-A
Chapel Hill, NC 27514

7b. ADDRESS *(City, State and ZIP Code)*:
Bolling Air Force Base
Washington, DC 20332

8a. NAME OF FUNDING/SPONSORING ORGANIZATION: AFOSR

8b. OFFICE SYMBOL *(If applicable)*:

9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER: F49620 85 C 0144

8c. ADDRESS *(City, State and ZIP Code)*:
Bolling Air Force Base
Washington, DC 20332

10. SOURCE OF FUNDING NOS.

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT NO.
61102F	2304	A5	

11. TITLE *(Include Security Classification)*: "Fluctuations near homogeneous states of chemical reactions with diffusion"

12. PERSONAL AUTHOR(S): Peter Kotelenez

13a. TYPE OF REPORT: technical

13b. TIME COVERED: FROM 9/1/85 TO 8/31/86

14. DATE OF REPORT *(Yr., Mo., Day)*: Nov. 1985

15. PAGE COUNT: 27

16. SUPPLEMENTARY NOTATION: Reaction-diffusion equation; stochastic model of nonlinear chemical reactions with diffusion; thermodynamic limit; central limit theorem; high density limit; stochastic partial differential equation

17. COSATI CODES

FIELD	GROUP	SUB. GR.

18. SUBJECT TERMS *(Continue on reverse if necessary and identify by block number)*:

19. ABSTRACT *(Continue on reverse if necessary and identify by block number)*

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20. DISTRIBUTION/AVAILABILITY OF ABSTRACT: UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS ☐

21. ABSTRACT SECURITY CLASSIFICATION: UNCLASSIFIED

22a. NAME OF RESPONSIBLE INDIVIDUAL: [redacted] Maj Woodruff

22b. TELEPHONE NUMBER *(Include Area Code)*: [redacted] (202) 767-5027

22c. OFFICE SYMBOL: NM

DD FORM 1473, 83 APR — EDITION OF 1 JAN 73 IS OBSOLETE.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

CENTER FOR STOCHASTIC PROCESSES

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FLUCTUATIONS NEAR HOMOGENEOUS STATES OF CHEMICAL REACTIONS WITH DIFFUSION

by

Peter Kotelenez

Technical Report No. 126

November 1985

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OF CHEMICAL REACTIONS WITH DIFFUSION

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Technical Report No. 122

November 1985

FLUCTUATIONS NEAR HOMOGENEOUS STATES OF CHEMICAL REACTIONS WITH DIFFUSION

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Abstract

Conditions are given under which a space-time jump Markov process describing the stochastic model of nonlinear chemical reactions with diffusion converges to the homogeneous state solution of the corresponding reaction-diffusion equation. The deviation is measured by a central limit theorem. This limit is a distribution valued Ornstein-Uhlenbeck process and can be represented as the mild solution of a certain stochastic partial differential equation.

REACTION-DIFFUSION EQUATION; STOCHASTIC MODEL OF NONLINEAR CHEMICAL REACTIONS WITH DIFFUSION; THERMODYNAMIC LIMIT; CENTRAL LIMIT THEOREM; HIGH DENSITY LIMIT; STOCHASTIC PARTIAL DIFFERENTIAL EQUATION

* This reasearch was done during the author's stay at the University of North Carolina at Chapel Hill in April 1985 and was supported by the Air Force Office of Scientific Research under AFOSR Grant No. F 49620 82 C 0009.

1. Introduction

Mathematical models of chemical reactions have been described by Gardiner McNeil, Walls and Matheson [12], Haken [15], Nicolis and Prigogine [31], van Kampen [20], and Arnold [1]. For the deterministic theory of reaction-diffusion equations we refer to Smoller [33] and the references therein.

In [3] Arnold and Theodosopulu have constructed a space-time jump Markov process $X_{v,N}$ by dividing a finite interval I (one-dimensional reactor) into N cells, counting the number of particles in each cell and dividing this number by a proportionality factor v (the cell size of an unscaled model). This density changes in each cell due to reaction and diffusion (which couples neighbouring cells). The rates by which this density changes are derived from an underlying partial differential equation (PDE). Under a high density assumption ($\frac{N^2}{v} \to 0$) Arnold and Theodosopulu (loc.cit.) derived the law of large numbers (LNN) in $L_2(I)$, i.e. $X_{v,N} \to X$ in $L_2(I)$, where X is the solution of the PDE. In Kotelenez [22], [25] the corresponding central limit theorem (CLT) was proved under the assumption that the reaction is linear. In this linear case the density could be taken low, because the LLN was proved in distribution spaces (cf. also Kotelenez [26]). On the other hand, nonlinear operations like multiplication are not defined on distributions (cf. Schwartz [32]). Therefore it seems to be convenient - if not necessary - to prove for nonlinear chemical reactions with diffusion the LLN in a function space by making a high density assumption (as in Arnold als Theodosopulu [3]) and then derive the CLT in a distribution space. This,however, causes certain numerical difficulties (cf. our Remark 3.1) which do not show up if we assume that the deterministic limit X is spatially homogeneous (cf. (2.1) and (2.5)). This assumption allows us to derive the LLN (Theorem 3.1) in a function norm and the CLT (Theorem 3.3) in a distribution norm. The limit Y and the CLT is a generalized Ornstein-Uhlenbeck process (if Y_0 is Gaussian) and can be represented as the mild

solution of a certain stochastic partial differential equation (SPDE). We describe the optimal (smoothest) state spaces for Y. Our main tool is the calculus of stochastic evolution equations as developed in Kotelenez ([21], [22], [24] - [27]) both for a fixed Hilbert state space and a nuclear Gel'fand triple (cf. (2.3)) as state space.

Apart from various Gaussian approximations to systems of (branching) Brownian motions (s. our references in Remark 2.2 - and also Kotelenez [27]) we would like to mention the diffusion approximations to spatially distributed neurons given in Walsh [36] and Kallianpur and Wolpert [19], where the limit is also a generalized Ornstein-Uhlenbeck process, which can be interpreted as the solution of a linear SPDE (as in our case).

Let us briefly describe the contents. In Section 2 we introduce both the deterministic and the stochastic models on an n-dimensional unit cube. In the first part on the deterministic model we introduce the nuclear Gel'fand triple (2.3) and prove that the linear operators from our models can be "nicely" defined on the Hilbert distribution spaces in (2.3). In the second part on the stochastic model we derive some bounds on $X_{V,N}$ and its martingale part. In Section 3 we prove the LLN in sup-norm (Theorem 3.1) with a certain speed of convergence. Then we describe the limiting Gaussian martingale part for the normalized martingale parts of $X_{V,N}$, prove in several steps the CLT and describe the limit (Theorem 3.3).

2. The Models

Following Arnold and Theodosopulu [3] and Arnold [1] we first introduce the (local) deterministic model, then construct the corresponding (local) stochastic model, and finally compare the two models.

2.1 The (local) deterministic model

Set $S := \{ q = (q_1, \ldots, q_n) \in \mathbb{R}^n : 0 \le q_i \le 1,\ i = 1, \ldots, n\}$. Let

$R(x) = b(x) - d(x) = \sum_{i=0}^{m} c_i x^i$ be a polynomial in $x \in \mathbb{R}$, where $c_o \geq 0$, $c_m < 0$ and $b(x)$ and $d(x)$ are polynomials of degree $\leq m$ with nonnegative coefficients. $\mathring{\Delta}$ denotes the Lapacian and $D > 0$ a diffusion coefficient. Then the concentration of one reactant with reflection at the boundary is given by the solution of the following PDE:

$$(2.1)\quad \begin{cases} \frac{\partial}{\partial t} X(t,q) = D\mathring{\Delta}X(t,q) + R(X(t,q)) \\ \frac{\partial}{\partial q_i} X(t,q) = 0, \quad q_i \in \{0,1\} \quad i = 1,\ldots,n \\ X_o(q) \geq 0 \end{cases}$$

Let $\mathbb{H}_o := L_2(S)$ be the Hilbert space of square integrable real valued functions on S equipped with the scalar product $\langle\varphi,\psi\rangle_o := \int_S \varphi(q)\psi(q)\,dq$, $\varphi,\psi \in \mathbb{H}_o$. In what follows we shall denote by $D\Delta$ the closure of $D\mathring{\Delta}$ w.r.t. the reflecting boundary conditions of (2.1). $D\Delta$ is self-adjoint nonpositive on $\mathbb{H}_o$ and has a discrete spectrum. Let $\ell = (\ell_1,\ldots,\ell_n)$ be a multiindex, where $\ell_i \in \mathbb{N} \cup \{0\}$, and set

$$\phi_{\ell_i} := \begin{cases} \sqrt{2}\, \cos \ell_i \pi(\cdot) & \ell_i \geq 1 \\ 1 & \ell_i = 0 \end{cases}$$

Then, the $\phi_\ell := \prod_{i=1}^{n} \phi_{\ell_i}$ are a complete orthonormal system (CONS) of eigenvectors of $D\Delta$ with eigenvalues $-D\mu_\ell := -D(\sum_{i=1}^{n} \ell_i^2 \pi^2)$. Consequently, the semigroup $T(t)$ generated by $D\Delta$ on $\mathbb{H}_o$ can be represented by

$$(2.2)\qquad T(t)\varphi = \sum_{\ell} e^{-D\mu_\ell t} \phi_\ell < \varphi,\phi_\ell >_o .$$

As in Kotelenez ([26], [27]) we introduce the nuclear Gel'fand triple determined by $D\Delta$

$$(2.3)\qquad \Phi \subset \mathbb{H}_\alpha \subset \mathbb{H}_o = \mathbb{H}_o' \subset \mathbb{H}_{-\alpha} \subset \Phi', \quad \alpha \geq 0.$$

In (2.3) we have $\mathbb{H}_\alpha := \mathrm{Dom}((I - D\Delta)^{\alpha/2})$, $\alpha \geq 0$ where I is the identity operator and "Dom" denotes "domain". $\mathbb{H}_\alpha$ is a real separable Hilbert space if equipped with the scalar product $<\cdot,\cdot>_\alpha := <(I-D\Delta)^{\alpha/2}\cdot,(I-D\Delta)^{\alpha/2}\cdot>_o$ (for the definition of the α-th power of a positive self-adjoint operator - cf. Yosida [38]). $\mathbb{H}'_o$, the strong dual of $\mathbb{H}_o$, is identified with $\mathbb{H}_o$, $\Phi = \bigcap_{\alpha\geq 0} \mathbb{H}_\alpha$ is a locally convex vector space whose topology is given by the set of norms $\{|\varphi|_\alpha := (<\varphi,\varphi>_\alpha)^{\frac{1}{2}}, \varphi \in \Phi\}$, and Φ' is the strong dual of Φ. $\mathbb{H}_{-\alpha}$ are those $\varphi' \in \Phi$ which can be extended to continuous functionals on $\mathbb{H}_\alpha$, $\alpha \geq 0$. $\mathbb{H}_{-\alpha}$ is a real separable Hilbert space with scalar product $<\cdot,\cdot>_{-\alpha}$, where for φ, $\psi \in \mathbb{H}_o$ $<\varphi,\psi>_{-\alpha} = <(I-D\Delta)^{-\alpha/2}\varphi, (I-D\Delta)^{-\alpha/2}\psi>_o$. Moreover, setting $\lambda_\ell := 1 + D\mu_\ell$, we obtain that

$$\phi_\ell^\alpha := \lambda_\ell^{-\alpha/2} \phi_\ell$$

is a CONS for $\mathbb{H}_\alpha$, $\alpha \in \mathbb{R}$. Hence

$$\mathbb{H}_\alpha = \{\varphi' \in \Phi' : \sum_\ell (\varphi',\phi_\ell)^2 \lambda_\ell^\alpha < \infty\},$$

where $(\cdot,\cdot)$ denotes the dual pairing. Thus, if we set

$$l_{2,\alpha} := \{(a_\ell) \in \mathbb{R}^\infty : \sum_\ell a_\ell^2 \lambda_\ell^\alpha < \infty\}$$

we see that (2.1) can be identified with a subset of $\mathbb{R}^\infty$, where $\mathbb{H}_\alpha$ is isomorphic to $l_{2,\alpha}$, $\alpha \in \mathbb{R}$. Clearly, the imbeddings in (2.3) are continuous and dense.

<u>Lemma 2.1</u>

For any $\alpha, \gamma \in \mathbb{R}$ s.t. $\alpha > \gamma + \frac{n}{2}$ the imbedding

$$\mathbb{H}_\alpha \to \mathbb{H}_\gamma$$

is Hilbert-Schmidt.

Proof

$$\sum_{\ell} |\phi_{\ell}^{\alpha}|_{\gamma}^{2} = \sum_{\ell} |\phi^{\alpha-\gamma}|_{o}^{2} < \infty \quad \text{iff}$$

$$\int_{1}^{\infty} \cdots \int_{1}^{\infty} (1+x_1^2 + \ldots + x_n^2)^{-\alpha+\gamma} \, dx_1 \ldots dx_n < \infty \quad \text{iff}$$

$$\int_{1}^{\infty} \cdots \int_{1}^{\infty} (1+x_1 \ldots + x_n)^{-2\alpha+2\gamma} \, dx_1 \ldots dx_n < \infty \quad \text{iff}$$

$$2\alpha > n + 2\gamma.$$

□

Since $(I-D\Delta)^{\alpha/2}$ and $T(t)$ commute, $T(t)$ can be extended (resp. restricted) to a strongly continuous semigroup $T_{\alpha}(t)$ on $\mathbb{H}_{\alpha}$, $\alpha \in \mathbb{R}$, s.t. for all $\alpha \in \mathbb{R}$

$$(2.4) \qquad |T_{\alpha}(t)|_{L(H_{\alpha})} = |T(t)|_{L(H_{o})} \leq 1.$$

$|\cdot|_{L(H_{\alpha})}$ denotes the usual operator norm on $\mathbb{H}_{\alpha}$, and the inequality in (2.4) holds because $D\Delta$ is dissipative (cf. Davies [9]). Let us denote by $D\Delta_{\alpha}$ the *generator* of $T_{\alpha}(t)$ (which is the extension (resp. restriction) of $D\Delta$). As in Kotelenez [26], Lemma 2.2, we obtain:

Lemma 2.2

For all $\alpha \in \mathbb{R}$, Dom $(D\Delta_{\alpha}) = \mathbb{H}_{\alpha+2}$ and $T_{\alpha}(t)$ is analytic.

For the rest of the paper we shall assume that the system (2.1) starts in a spatially homogeneous state $X_o = \rho_o > 0$. This implies that the solution $X(t,r)$ of (2.1) is spatially homogeneous, i.e. $X(t,r) \equiv \rho(t)$ satisfies the ordinary differential equation

$$(2.5) \qquad \frac{d}{dt}\rho(t) = R(\rho(t)), \quad \rho(0) = \rho_o > 0$$

(cf. Arnold [1]). (2.5) has a unique positive bounded solution which

is strictly positive for all $t \geq 0$ (cf. Coddington and Levinson [5]).

2.2 The (local) stochastic model

We cover S with grid of N n-dimensional cubes (cells) of size h^n which are parallel to the axes. The cell corresponding to the grid point r^j is defined by

$$[r^j) := \{r \in S: r_i^j \leq r_i < r_i^j + h,\ i = 1, \ldots, n\}\ ,\quad j = 1, \ldots, N.$$

Let v be a parameter (which is explained in Remark 2.1) and denote by $\mathbb{E}_N$ the (countable) state space of elements $k = (k_{r^j})_{([r^j)\subset S)}$, where $k_{r^j} \in \frac{\mathbb{N}}{v} \cup \{0\}$. Set

$$\mathbb{H}_{o,N} := \{\varphi \in \mathbb{H}_o : \varphi \text{ constant on each } [r^j)\ \}.$$

Then

$$\mathbb{E}_N \subset \mathbb{H}_{o,N}$$

and

$$\pi_N := \mathbb{H}_o \rightarrow \mathbb{H}_{o,N} \quad \text{defined by}$$

$$\pi_N\varphi(r) := \varphi_N(r) := h^{-n}\int_{[r^j)} \varphi := h^{-n}\int_{[r^j)} \varphi(q)dq \quad \text{if} \quad r \in [r^j)$$

is a projection from $\mathbb{H}_o$ onto $\mathbb{H}_{o,N}$.

Now we define a Markov chain on $\mathbb{E}_N$ through the Q-matrix of its transition intensities:

$$(2.6)\quad \beta(k,m) := \begin{cases} vb(k_{r^j}) & m = k + e_{r^j} =: m_{+o} \\ vd(k_{r^j}) & m = k - e_{r^j} =: m_{-o} \\ vDh^{-2}k_{r^j} & m = k + e_{r^j\pm h_i} - e_{r_j} =: m_{\pm i},\ i = 1,\ldots,n \\ & \quad \text{for } 0 \leq r^j \pm h_i \leq 1 \\ -\sum_{i=o}^{n} \beta(k,m_{\pm i}) & m = k \\ 0 & \text{otherwise.} \end{cases}$$

Here $e_{rj} = (1_{[r^j)})$, where $1_{[r^j)}(q) = 1$ if $q \in [r^j)$, $= 0$ otherwise , and $h_i = (0,..,0,h,0,..,0)$ where all but the i-th coordinate are zero. Hence, we obtain the distributions $P(t,k)$ determined by $Q = (\beta(k,m)_{k,m\in \mathbb{E}_N})$ as the unique solution of Kolmogorov's backward equation (which is called in the application-oriented literature the "multivariate Master equation) (cf. Arnold [1]). The corresponding (canonical) cadlag Markov process will be denoted by

$$(2.7)\quad \begin{cases} X_{v,N} & \text{(generated by } Q) \\ X_{v,N}(0) = X_{v,N,o} \in \mathbb{E}_N & \text{a given initial distribution.} \end{cases}$$

In what follows we shall assume that the stochastic basis for $X_{v,N}$ $(\Omega, F, F_{v,N,t}, P)$ is complete with right continuous filtration.

Remark 2.1

We can view $X_{v,N}$ as the rescaled density Markov process of Arnold [1] and Arnold and Theododopulu [3] on a cube of volume $vN = V$ with cells of size v, where the number of particles is proportionel to v.

Remark 2.2

If $b(r) = br + c_o$, $d(r) = dr$, for some constants $b,d > 0$ then $X_{v,N}$ is a branching diffusion with immigration (c_o) on the grid. This case was investigated in Kotelenez [25], [26], and the limit theorems therein corresponded to limit theorems for branching Brownian motions obtained by Holley and Stroock [16], Gorostiza [14] and, in the absence of branching ($b = d = c_o = 0$) to Martin-Löf [30] and Itô [18] (cf. also Walsh [37]). For a diffusion approximation to branching Brownian motions - cf. Dawson [11].

In what follows we shall not explicitly write the parameter v, i.e., we shall write X_N instead of $X_{v,N}$ etc.

Extend $\varphi_N \in \mathbb{H}_{o,N}$ by reflection to

$$S_h := \{r \in \mathbb{R}^n : -h \le r_i \le 1+h,\ i = 1,\dots,n\}$$

and set

$$\nabla^N_{\pm i}\, \varphi_N \quad := \quad h^{-1}[\varphi_N(r \pm h_i) - \varphi_N(r)]$$

$$\Delta_N \varphi_N \quad := \quad -\sum_{i=1}^{n} \nabla^N_{-i}\ \nabla^N_i\ \varphi_N.$$

Remark 2.3

In view of our boundary condition we easily see that $D\Delta_N$ is selfadjoint and dissipative both as an operator on $\mathbb{H}_{o,N}$ and $\mathbb{H}_o$ (where on $\mathbb{H}_o$ Δ_N is defined by $\Delta_N \circ \pi_N$). If we set

$$\bar{\phi}_{\ell_i,N} \quad := \quad \frac{\phi_{\ell_i,N}}{|\phi_{\ell_i,N}|_o}$$

we see that

$$\{\bar{\phi}_{\ell,N} := \prod_{i=1}^{n} \bar{\phi}_{\ell_i,N}\ ,\quad \ell_i < h^{-1}\}$$

is a CONS of eigenvectors of $D\Delta_N$ for $\mathbb{H}_{o,N}$ with eigenvalues

$$D\mu_{\ell,N} \quad := 2^n D N^2 \prod_{i=1}^{n} \{1 - \cos \ell_i h\pi\}.$$

The waiting time parameter for X_N is given by

$$\sigma_N(k) = v \sum_{r^j \in S} |R|(k_{r^j}) + h^{-2} \sum_{i=1}^{n} 2Dk_{r^j}$$

with $|R|(x) = b(x) + d(x)$, $x \in \mathbb{R}$. Hence, if

$$\Theta_N(k,m) := (\sigma_N(k))^{-1}\ \beta(k,m)$$

denotes the jump distribution function ($\beta(k,m)$ from (2.6)), then the infinitesimal generator for X_N is given by

$$(2.8)\quad (Af)(k) = \sigma_N(k) \int_{\mathbb{E}_N} [f(m) - f(k)]\, \Theta_N(k,dm),$$

where $f\colon \mathbb{E}_N \to \mathbb{R}$ is bounded and measurable (Gihman and Skorohod [13]).

Let

$$|||\, \phi_N \,||| := \sup_{r \in S} |\phi_N(r)|$$

be the sup-norm on $\mathbb{H}_{o,N}$. If there is a finite constant $K(v,N)$ s.t.

$$(2.9)\qquad |||\, X_{v,N}(0)\,||| \le K(v,N) \quad \text{a.s.}$$

then by a lemma of Kurtz [28] (cf. Arnold and Theodosopulu [3] and Kotelenez [25])

$$(2.10)\quad \begin{cases} Z_N(t) := X_N(t) - X_N(0) - \int_0^t \sigma_N(X_N(s)) \int_{\mathbb{E}_N} (z - X_N(s))\Theta_N(X_N(s),dz)ds \\ \qquad = X_N(t) - X_N(0) - \int_0^t [D\Delta_N X_N(s) + R(X_N(s))]ds \end{cases}$$

is an $\mathbb{H}_{o,N}$ -valued square integrable cadlag martingale.

We shall assume (2.9) throughout the paper.

Hence, X_N satisfies formally the stochastic evolution eqaution

$$(2.11)\quad \begin{cases} dX_N(t) = [D\Delta_N X_N(t) + R(X_N(t))]dt + dZ_N(t) \\ X_N(0) = X_{N,o} \end{cases},$$

and the difference $X_N(t) - X(t)$, where X is the solution of (2.1)/(2.5), satisfies

$$(2.12)\begin{cases} X_N(t) - X(t) = X_N(0) - X(0) + \int_0^t (D\Delta_N + R'(X(s))(X_N(s) - X(s))ds \\ \qquad + \int_0^t (X_N(s) - X(s))^2 \tilde{R}(X_N(s),X(s))ds \\ \qquad + Z_N(t) \quad , \end{cases}$$

$R'(x)$ is the derivative of $R(x)$, $\tilde{R}(y,x)$ is a polynomial in y and x of degree $\le m-2$, and $R'(X(s))$ and $(X_N(s) - X(s))^2$ are interpreted as multiplication operators.

Note that both $D\Delta_N + R'(X(s))$ and $D\Delta + R'(X(s))$ are quasi-generators of evolution operators $U_N(t,s)$ and $U(t,s)$ on $\mathbb{H}_{o,N}$ and $\mathbb{H}_o$, respectively. (For the definition of evolution operators $V(t,s)$, i.e., strongly continuous two-parameter semigroups - cf. Curtain and Pritchard [7] and Tanabe [34], where $V(t,s)$ is called fundamental solution - any strongly continuous one-parameter semigroup is, of course, also an evolution operator.) Consequently, by variation of constants, (2.12) yields

$$(2.13)\quad \begin{aligned} X_N(t) - X(t) &= U_N(t,0)(X_N(0) - X(0)) + \int_0^t U_N(t,s)dZ_N(s) \\ &+ \int_0^t U_N(t,s)(X_N(s) - X(s))^2 \tilde{R}(X_N(s),X(s))ds. \end{aligned}$$

In order to give a meaning to the stochastic convolution integral in (2.13) we recall from Kotelenez [21], [24]:

<u>Definition 2.1</u>

Let $\mathbb{H}$ be a separable Hilbert-space with Hilbert space norm $|\cdot|_H$ and $V(t,s)$ an evolution operator on $\mathbb{H}$, $0 \le s \le t < \infty$. $V(t,s)$ is of contraction-type or, equivalently, $V(t,s) \in G(1,\beta_{\hat{t}})$ if for all $\hat{t} > 0$ there is a finite constant $\beta_{\hat{t}} \ge 0$ s.t.

$$(2.14)\qquad |V(t,s)|_{L(\mathbb{H})} \le e^{\beta_{\hat{t}}(t-s)}$$

for all $0 \le s \le t \le \hat{t}$.

Remark 2.4

Let M be an $\mathbb{H}$-valued locally square integrable cadlag martingale and $V(t,s) \in G(1,\beta_{\hat{t}})$ on $\mathbb{H}$. Then, from Kotelenez [21], we have

(i) If M is cadlag, then $\int_0^{\cdot} V(\cdot,s)dM(s)$ has a cadlag version; if M is continuous, then $\int_0^{\cdot} V(\cdot,s)dM(s)$ has a continuous version.

A partial result from Kotelenez [24] is the following:

(ii) If $V(t,s)$ has a quasi-generator $A(t)$ and $\mathrm{Dom}(A(t))$ is independent of t then for all $\hat{t} > 0$ there is a finite constant $c = c(\hat{t},\beta)$ depending only on the scalar product $\langle \cdot,\cdot \rangle_{\mathbb{H}}$, $\hat{t}$ and β s.t. for all $t \le \hat{t}$

$$(2.15) \qquad E \sup_{0\le s\le t} \left|\int_0^s V(s,u)dM(u)\right|_{\mathbb{H}}^2 \le c\, e^{4\beta_{\hat{t}} t}\, E|M(t)|_{\mathbb{H}}^2 .$$

For more general properties and inequalities for stochastic convolution integrals cf. Kotelenez [24], [27].

Since $X(t)$ is constant in the space variable (cf. (2.5)) we obtain

$$(2.16) \qquad \begin{cases} U_N(t,s) = T_N(t-s)\exp\left(\int_s^t R'(X(u))du\right) \\ U_\alpha(t,s) = T_\alpha(t-s)\exp\left(\int_s^t R'(X(u))du\right), \quad \alpha \in \mathbb{R}, \end{cases}$$

where the last equation means that $U(t,s)$ is extendible (resp. restrictable) to the $\mathbb{H}_\alpha$. Let us denote by $(\mathbb{H}_{\alpha,N}, \langle \cdot,\cdot \rangle_\alpha)$ $\mathbb{H}_{0,N}$ equipped with the Hilbert norm $\langle \cdot,\cdot \rangle_\alpha$, $\alpha \in \mathbb{R}$. Set

$$\beta := \sup_{0 \le t < \infty} R'(X(t))$$

and note that $\beta < \infty$ by our assumption on $R(x)$ (cf. (2.5)).

Lemma 2.3

For all $\alpha \in \mathbb{R}$

$$U_\alpha(t,s) \in G(1,\beta) \quad \text{on } \mathbb{H}_\alpha$$

$$U_N(t,s) \in G(1,\beta) \quad \text{on } \mathbb{H}_{\alpha,N} .$$

Proof

(i) The statement for $U_\alpha(t,s)$ follows from (2.4) and (2.16).

(ii) Let $x \in \mathbb{H}_{o,N}$. Then

$$\begin{aligned} |U_N(t,s)x|_\alpha^2 &= \sum_\ell < U_N(t,s)x, \phi_\ell >_o^2 \lambda_\ell^\alpha \\ &= \sum_\ell < x, U_N(t,s)\phi_{\ell,N} >_o^2 \lambda_\ell^\alpha \\ &\le \sum_\ell e^{2\beta(t-s)} < x, T_N(t-s)\phi_{\ell,N} >_o^2 \lambda_\ell^\alpha \\ &\qquad\qquad \text{by (2.16)} \\ &\le e^{2\beta(t-s)} |x|_{-\alpha}^2 \quad , \end{aligned}$$

since $T_N(t)$ is a contraction and $\phi_{\ell,N}$ is an eigenvector of $T_N(t)$.

□

The previous considerations show that the limit behaviour of $X_N(t) - X(t)$ essentially depends on the limit behaviour of $Z_N(t)$, $U_N(t,s)$ and the last term in (2.13).

We shall first give an estimate on the variance of $Z_N(t)$. To this end we define an operator on $\mathbb{H}_o$ by

$$(2.17) \qquad F_N(\varphi) := D \sum_{i=1}^{n} \nabla_{-i}^N \, \varphi \nabla_i^N + \nabla_i^N \, \varphi \nabla_{-i}^N \; + |R|(\varphi)$$

where $\varphi \in \mathbb{H}_o$ and $|R|(\varphi)$ act as multiplication operators. As in the linear case of Kotelenez [25] we obtain

<u>Lemma 2.4</u>

For arbitrary $\varphi \in \mathbb{H}_o$

$$(2.18) \qquad E < Z_N(t),\varphi >_o^2 = \frac{1}{vN} \int_o^t E < F_N(X_N(s))\varphi_N,\varphi_N >_o ds.$$

We need estimates on $X_N(t,r)$ which satisfies by variation of constants

$$(2.19) \qquad X_N(t) = T_N(t)X_N(0) + \int_o^t T_N(t-s)dZ_N(s) + \int_o^t T_N(t-s)R(X_N(s))ds.$$

Set $\qquad \bar{\rho} := \max \{R(x) : x \in \mathbb{R}_+\}.$

The definition of $R(x)$ implies $\bar{\rho} < \infty$.

<u>Lemma 2.5</u>

For any $t > 0$

$$(2.20) \qquad \sup_{o \le s \le t} ||| E X_N(s) ||| \le t\bar{\rho} + ||| E X_N(0) ||| ,$$

$$(2.21) \qquad ||| X_N(t) ||| \le t\bar{\rho} + ||| X_N(0) ||| + \sqrt{N} \; \left| \int_o^t T_N(t-s)dZ_N(s) \right|_o .$$

<u>Proof</u>

(i) From Kotelenez [22], Lemma A.7 and Davies [9], Th. 7.16 we obtain that $T_N(t)$ is positivity-preserving on $\mathbb{H}_{o,N}$, i.e., leaves the cone of nonnegative functions invariant, which implies (2.20).

(ii) We easily check that for any $\varphi_N \in \mathbb{H}_{o,N}$

(2.22) $$|||\varphi_N||| \leq \sqrt{N}\ |\varphi_N|_o ,$$

whence we obtain (2.21) from (2.19).

□

3. Limit Theorems

Theorem 3.1 (LLN)

Assume (2.9) in addition to

(I) $v = N^p$, where $p > \dfrac{2\gamma+1+\frac{2}{n}}{1-2\gamma}$ and $\gamma \in [\frac{1}{4}, \frac{1}{2})$ arbitrary and fixed ;

(II) $E\,|||(vN)^\gamma(X_N(0) - X(0))||| \to 0$, as $N \to \infty$.

Then for all $\hat{t} > 0$, $\delta > 0$

$$P\{\sup_{0\leq t\leq \hat{t}} |||(vN)^\gamma(X_N(t) - X(t)|||\ > \delta\} \to 0; \text{ as } N \to \infty .$$

Proof

(i) (2.18), (2.20) and our assumptions imply the existence of a finite constant K s.t. for any $t \geq 0$

(3.1) $$E\,|Z_N(t)|_o^2 \leq \frac{K(t+1)}{v}(c_o + N^{2/n}) ,$$

where by (2.15) and Lemma 2.3 there is for any $\hat{t} \geq 0$ a finite constant $K(\beta,\hat{t})$ s.t. for $\gamma \in [\frac{1}{4}, \frac{1}{2})$ and

$$\eta_N(t) := (vN)^\gamma N^{1/2} \max\{|\int_0^t U_N(t,s)dZ_N(s)|_o ,|\int_0^t T_N(t-s)dZ_N(s)|_o\}$$

$$E \sup_{0\le t\le \hat t} \eta_N^2(t) \le K(\beta,\hat t)\, \frac{(vN)^{2\gamma}}{v}\, N(c_0 + N^{2/n})$$

$$= K(\beta,\hat t) N^{p(2\gamma-1) + 2/n + 1 + 2\gamma}$$

$$\to 0, \text{ as } N \to \infty \text{ by the definition of } p.$$

(ii) (2.16) implies that $U_N(t,s)$ is positivity preserving since $T_N(t)$ is positivity-preserving. Abbreviating

$$\zeta_N(t) := e^{\beta t} (vN)^{\gamma} ||| X_N(0) - X(0) ||| + \eta_N(t)$$

and

$$\psi_N(t) := e^{\beta \hat t} ||| \tilde{R}(X_N(t), X(t) ||| \cdot ||| X_N(t) - X(t) ||| , \ t \le \hat t$$

the Gronwall-Bellmann lemma and (2.13) imply

$$(vN)^{\gamma} ||| X_N(t) - X(t) ||| \le \zeta_N(t) + \int_0^t \zeta_N(s)\psi_N(s)\exp\left(\int_0^t \psi_N(u)\,du\right)ds, \quad t \le \hat t.$$

Since by step (i) $\sup_{0\le t\le \hat t} \psi_N(t)$ is stochastically bounded as $N \to \infty$ and $\sup_{0\le t\le \hat t} \zeta_N(t)$ tends to zero in mean square the proof is finished.

□

Set

$$M_N := (vN)^{1/2} Z_N$$

and define for $\varphi \in \mathbb{H}_1$ the continuous analogue to (2.17):

$$\text{(3.2)} \qquad F(\varphi) := -2D \sum_{i=1}^{n} \partial_i \varphi\, \partial_i + |R|(\varphi)$$

where again φ and $|R|(\varphi)$ act as multiplication operators. Denote for $\mu \in (0,1)$ by

$$C^{\mu}([0,\infty); \mathbb{H})$$

the space of Hölder continuous $\mathbb{H}$-valued functions with Hölder exponent μ, where $\mathbb{H}$ is some Hilbert space.

Lemma 3.1

There is a unique (in distribution) Φ'-valued Gaussian martingale M on some probability space $(\tilde{\Omega},\tilde{F},\tilde{F}_t,\tilde{P})$ with characteristic functional

(3.3) $$\tilde{E}\exp(i(M(t),\varphi)) = \exp(-\frac{1}{2}\int_0^t < F(X(s))\varphi,\varphi >_0 ds),$$

$\varphi \in \Phi$, where $\tilde{E}$ is the mathematical expectation w.r.t. $\tilde{P}$. Moreover, for any $\alpha > \frac{n}{2}+1$ and any $\mu \in (0,\frac{1}{2})$

$$M \in C^{\mu}([0,\infty),\mathbb{H}_{-\alpha}) \quad \text{a.s.}$$

The proof of the existence and uniqueness is given in Itô [17](cf. also Ustunel [35]), and the Hölder continuity follows from Kotelenez [23].

Since X(t) is spatially homogeneous and strictly positive we easily check that F(X(t)) as a positive self-adjoint operator on $\mathbb{H}_0$ is just equal to $-2X(t)D\Delta + |R|(X(t))$, which has the same eigenfunctions ϕ_ℓ as $D\Delta$. Thus, the square root of F(X(t)) can be considered as an element $F_\alpha^{1/2}(X(t))$ from $L(\mathbb{H}_\alpha,\mathbb{H}_{\alpha-1})$ for all $\alpha \in \mathbb{R}$. If $\alpha > \frac{n}{2}$ then there is an $\mathbb{H}_{-\alpha}$ valued Wiener process W(t) on $\mathbb{H}_{-\alpha}$ which is the cylindrical Brownian motion on $\mathbb{H}_0$ (cf. Itô [17]). We may without loss of generality assume that W(t) is also defined on $(\tilde{\Omega},\tilde{F},\tilde{F}_t,\tilde{P})$ from Lemma 3.1. Repeating now the proof of Lemma 2.4 in Kotelenez [25] we obtain

Lemma 3.2

(3.4) $$M \overset{\mathcal{D}}{=} \int_0^{\cdot} F_{-\alpha+1}^{1/2}(X(s)dW(s) \quad \text{(equal in distribution)}$$

on $C^{\mu}([0,\infty);\mathbb{H}_{-\alpha-1})$ for all $\alpha > \frac{n}{2}$, all $\mu \in (0,\frac{1}{2})$.

Let us denote by $D([0,\infty);\mathbb{H})$ the complete metric space of $\mathbb{H}$-valued cadlag functions, where $\mathbb{H}$ is a separable Hilbert space (i.e. the Skorohod space - cf. Billingsley [4] and Kurtz [29]) and by "$\Rightarrow$" weak convergence.

Lemma 3.3

Under the assumptions of Theorem 3.1 for all $\alpha > \frac{n}{2} + 1$

$$M_N \Rightarrow M \text{ on } D([0,\infty);\mathbb{H}_{-\alpha}),$$

where M is the Gaussian martingale given in Lemma 3.1.

Proof

(i) The weak convergence of $M_N(t)$ to $M(t)$ for fixed t follows as in the linear case (cf. Kotelenez [25], [26]).

(ii) We shall estimate the "modules of continuity". Set for some (large) $K > 0$

$$\tau_N := \inf\{t \geq 0 : |||X_N(t)||| \geq K\}$$

and
$$\tilde{M}_N(t) := M_N(t\wedge\tau_N)$$

where "$\wedge$" denotes "min". Then, abbreviating $F_{N,t} := \sigma(X_N(s), s\leq t)$ we obtain for $t \leq \hat{t}$, $s > 0$

$$E\,\{|M_N(t+s) - M_N(t)|^2_{-\alpha} \wedge 1|\, F_{N,t}\}$$

$$(*) \qquad \leq E\,\{|\tilde{M}_N(t+s) - \tilde{M}_N(t)|^2_{-\alpha} \mid F_{N,t}\}$$

$$+ E\,\{1_{\{\tau_N < \hat{t}+s\}} \mid F_{N,t}\}\ .$$

Take the CONS $\{\varphi^{\alpha}_{\ell}\}$ for $\mathbb{H}_{\alpha}$. By (2.18) (cf. Kurtz [28] and Kotelenez [22], [25] for the step from the unconditional to the conditional expectation) the first term in the r.h.s. of (*) can be estimated from above by

$$(**)\begin{cases} \sum_{\ell} E\{\int_t^{t+s} < \phi_\ell^\alpha, F_N(X_N(u\wedge\tau_N))\phi_\ell^\alpha >_o du \mid F_{N,t}\} \\ \leq 2\sum_\ell E\{\int_t^{t+s} <(\phi_\ell^\alpha)^2, |R|(X_N(u\wedge\tau_N))>_o + \sum_{i=1}^n D<\partial_i\phi_\ell^\alpha)^2, X_N(u\wedge\tau_N)>_o du \mid F_{N,t}\} \end{cases}$$

by Lemma A.2 in Kotelenez [22]

$$\leq \tilde{K}s$$

for some $\tilde{K} < \infty$ since by Lemma 2.1 $\mathbb{H}_{\alpha-1} \hookrightarrow \mathbb{H}_o$ is Hilbert-Schmidt and $|||\, X_N(t\wedge\tau_N)\, ||| \leq K + 1 < \infty$.

Setting

$$Y_{N,\hat{t}}(s) := \tilde{K}s + 1_{\{\tau_N < \hat{t}+s\}}$$

we obtain from Theorem 3.1

$$\lim_{s\to o} \overline{\lim_{N\to\infty}}\, E\, Y_{N,\hat{t}}(s) = 0.$$

(iii) (i) and (ii) imply by Theorem 2.7 of Kurtz [29] the weak convergence of M_N to M.

□

Theorem 3.2

Under the assumptions of Theorem 3.1 for all $\alpha > \frac{n}{2} + 1$

$$\int_o^{\cdot} U_N(\cdot,s)dM_N(s) \Rightarrow \int_o^{\cdot} U_{-\alpha}(\cdot,s)dM(s) \quad \text{on } D([0,\infty);\mathbb{H}_{-\alpha}).$$

Proof

(i) Let d_p denote the Prohorov metric on $D([0,\infty);\mathbb{H}_{-\alpha})$ (cf. Billingsley [4]). Let π_k be the projection of $\mathbb{H}_{-\alpha}$ onto $L(\phi_\ell : \ell_i < k$ for all $i = 1,..,n)$ (the linear hull spanned by those ϕ_ℓ whose multiindices

$\ell = (\ell_1,\ldots,\ell_n)$ satisfy $\ell_i < k$ for all $i = 1,\ldots,n$). The corresponding projection from $\mathbb{H}_{o,N}$ onto $L(\phi_{\ell,N}: \ell_i < k$ for all $i = 1,\ldots,n)$ if $k < h^{-1}$ will be denoted by p_k^N. If $k \geq h^{-1}$ then we set $p_k^N \mathbb{H}_{o,N} = \mathbb{H}_{o,N}$. Set $\pi_k^{\perp} = I - \pi_k$ and $p_k^{\perp N} = I - p_k^N$, where I denotes the identity operator on the corresponding spaces.

(ii) Abbreviating the convolution integrals $\int_o^{\cdot} U_N(\cdot,s)dM_N(s)$ by $\int U_N dM_N$ etc. we obtain

$$
(*)\qquad
\begin{aligned}
& d_p(\textstyle\int U_N dM_N, \int U dM) \\
&\leq d_p(\textstyle\int U_N dM_N, \int U_N p_k^N dM_N) \\
&\quad + d_p(\textstyle\int U_N p_k dM_N, \int U\pi_k dM) \\
&\quad + d_p(\textstyle\int U\pi_k dM, \int U dM).
\end{aligned}
$$

(iii) By Lemma 2.3 and (2.15) for any $\hat{t} \geq 0$

$$
\begin{aligned}
& E \sup_{o\leq t\leq \hat{t}} \left|\int_o^t U_N(t,s) p_k^{\perp N} dM_N\right|_{-\alpha}^2 \\
&\leq c\, e^{4\beta\hat{t}}\, E|p_k^{\perp N} M_N(\hat{t})|_{-\alpha}^2 \\
&\leq c\, e^{4\beta\hat{t}} \sum_{\ell_i \geq k} E < M_N(\hat{t}), \phi_{\ell N} >_o^2 \lambda_\ell^{-\alpha} \\
&\leq c\, e^{4\beta\hat{t}}\, K(\hat{t}+1) \sum_{\ell_i\geq k} \lambda_\ell^{-\alpha+1}
\end{aligned}
$$

as in the proof of Lemma 3.3 for some finite constant K.Since $\sum_\ell \lambda_\ell^{-\alpha+1} < \infty$ by Lemma 2.1 the r.h.s. of the last inequality can be made arbitrarily small by choosing k large. Hence for given $\varepsilon > 0$ there is a $k_1(\varepsilon)$ s.t. for all $k \geq k_1(\varepsilon)$ and all N

$$d_P(\int U_N dM_N, \int U_N p_k^N dM_N) \le \frac{\varepsilon}{3}$$

(cf. Kotelenez [24], [25]).

The third term in (*) can be estimated in the same way. The second term in (*) tends to zero for fixed k. Indeed, by partial integration $\int_0^t U_N(t,s) p_k^N dM^N(s) = p_k^N M^N(t) + \int_0^t U_N(t,s)[D\Delta_N + R'(X(s))] p_k^N M_N(s) ds$ and $\int_0^t U(t,s)\pi_k dM(s) = \pi_k M(t) + \int_0^t U(t,s) \ (D\Delta + R'(X(s))\pi_k M(s) ds$. Hence, the Trotter-Kato theorem (Davies [9], Theorem 3.17, Kotelenez [24], Remark 4.1, and Kotelenez [22], Lemmas A.1, A.3) and the definition of p_k^N and π_k imply the conditions of Theorem 5.5 in Billingsley [4], Ch.I.

(iv) Since weak convergence on $D([0,\infty); \mathbb{H}_{-\alpha})$ and convergence w.r.t. the Prohorov metric d_P are equivalent (cf. Kurtz [29] and Billingsley [4], Appendix III, Th. 5) the proof is finished.

□

Fix

$$\alpha > \frac{n}{2} + 1 .$$

Let Y_o be an $\mathbb{H}_{-\alpha+1}$-valued square integrable random variable on $(\tilde{\Omega}, \tilde{F}, \tilde{F}_t, \tilde{P})$ independent of $W(t)$ for all $t \ge 0$ and $\underline{Y}_o$ a square integrable $\mathbb{H}_{-\alpha+1}$-valued random variable on (Ω, F, F_t, P) such $\underline{Y}_o \stackrel{\mathcal{D}}{=} Y_o$. Further, let C denote an arbitrary $\int U\, F_{-\alpha+1}^{1/2}\, dW$ - continuity set of $D([0,\infty); \mathbb{H}_{-\alpha})$ (cf. Billingsley [4]) and E an arbitrary element from $\sigma(Y_o)$. We make the following asymptotic independence assumption:

$$(3.5)\qquad \begin{cases} Y_{N,o} := (vN)^{1/2}(X_{N,o} - X_o) \to Y_o \text{ in probability on } \mathbb{H}_{-\alpha+1} \\ P\{(\int U_N dM_N \in C) \cap E\} \to \tilde{P}\{\int U F_{-\alpha+1}^{1/2} dW \in C\}\tilde{P}[E) \end{cases}$$

(cf. Billingsley [4], Ch.I, Th. 2.1 - The second condition in (3.5) is, e.g., satisfied if M_N is independent of $\sigma(\underline{Y}_o)$ for all (large) N.

Let δ denote the Fréchet derivative, $B([0,\hat{t}] \times \mathbb{H}_{-\alpha})$ the real valued measurable functions g with domain $[0,\hat{t}] \times \mathbb{H}_{-\alpha}$ s.t. $\frac{\partial g}{\partial t}$, $\delta g, \delta^2 g$, and $D\Delta_{-\alpha}\delta g$ exist, are continuous in x and t, and uniformly bounded in norm on $[0,T] \times \mathbb{H}_{-\alpha}$. $Q^{1/2}$ is the square root of the covariance operator of W(t) on $\mathbb{H}_{-\alpha+1}$ and $F^{1/2*}_{-\alpha+1}(X(t))$ is the dual operator of $F^{1/2}_{-\alpha+1}(X(t))$ (after identifying the duals of $\mathbb{H}_{-\alpha}$ and $\mathbb{H}_{-\alpha+1}$ with $\mathbb{H}_{-\alpha}$ and $\mathbb{H}_{-\alpha+1}$, respectively. Finally, "Tr" denotes "trace".

Now we can state our final result under the assumptions of the LLN.

<u>Theorem 3.3</u> (CLT)

Assume (2.9) and (3.5) for fixed $\alpha > \frac{n}{2} + 1$ in addition to

(I) $\quad v = N^p$ where $p > \dfrac{2\gamma+1+\frac{2}{n}}{1-2\gamma}$ and $\gamma \in [\frac{1}{4}, \frac{1}{2})$ arbitrary and fixed ;

(II) $\quad E\,|(vN)^{\gamma}(X_N(0) - X(0))|_o^2 \to 0$, as $N \to \infty$.

Then for $Y_N := (vN)^{1/2}(X_N - X)$

(i) $\quad Y_N \Rightarrow Y$ on $D([0,\infty); \mathbb{H}_{-\alpha})$

where

$$(3.6) \qquad Y(t) = U_{-\alpha}(t,0)Y_o + \int_0^t U_{-\alpha}(t,s)\, F^{1/2}_{-\alpha+1}(X(s))dW(s)$$

is the mild solution of the stochastic partial differential equation

$$(3.7) \qquad \begin{cases} dY(t) = [D\Delta_{-\alpha} + R'(X(t))]Y(t)dt + F^{1/2}_{-\alpha+1}(X(t))dW(t) \\ Y(0) = Y_o \end{cases}$$

(ii)

(3.8) $$Y \in \begin{cases} C^{\mu}([0,\hat{t}];\, \mathbb{H}_{-\alpha}) & \text{a.s. for all } \mu < \frac{1}{2}\,,\ \text{all } \hat{t} > 0 \\ C([0,\hat{t}];\, \mathbb{H}_{-\alpha+1}) & \text{a.s. for all } \hat{t} > 0. \end{cases}$$

and $Y(t)$, $t > 0$, does not define a σ-additive measure on $\mathbb{H}_{-\alpha}$ for $\gamma \le \frac{n}{2}$, i.e., the second relation in (3.8) is the maximal regularity of Y on the Hilbert scale (2.3).

(iii) Y is a Markov process, and its weak generator is given by

(3.9) $$\begin{aligned} A(t)g(t,\varphi') &= \frac{\partial}{\partial t}\, g(t,\varphi') + \langle [D\Delta_{-\alpha} + R'(X(t))]\delta g(t,\varphi'),\varphi'\rangle_{-\alpha} \\ &\quad + \frac{1}{2}\,\mathrm{Tr}\{Q^{1/2}\, F^{1/2*}_{-\alpha+1}(X(t))\delta^2 g(t,\varphi')F^{1/2}_{-\alpha+1}(X(t))Q^{1/2}\}, \end{aligned}$$

where $g \in B([0,\hat{t}] \times \mathbb{H}_{-\alpha})$.

<u>Proof</u>

(i) The norm of the normalized last term in (2.13) can be estimated as follows:

$$\begin{aligned} &\left| \int_0^t U_N(t,s)(X_N(s) - X(s))^2 (\nu N)^{1/2}\, \tilde{R}(X_N(s),X(s)) \right|_{-\alpha} \\ &\le \int_0^t e^{\beta(t-s)}\, |||\, (\nu N)^{1/4}(X_N(s)-X(s))\, |||^2 \cdot |||\, \tilde{R}(X_N(s',X(s)\, |||\, ds \;\to 0, \end{aligned}$$

as $N \to \infty$ uniformly on compact intervals in probability by Theorem 3.1 and the stochastic boundedness of $|||\, \tilde{R}(X_N(s),X(s)\, |||$ (cf. the proof of Theorem 3.1). Therefore, the weak convergence of Y_N to Y follows from Theorem 3.2 and our assumptions as in Kotelenez [25], [26].

(ii) The Hölder continuity follows from DaPrato, Iannelli and Tubaro [8] and Kotelenez [23].

The spatial regularity follows from the estimate

$$|\int_s^t U_{-\alpha}(t,u)F^{1/2}_{-\alpha+1}(X(u))dW(u)|_{-\alpha+1} \le |\int_s^t T_{-\alpha}(t-u)F^{1/2}_{-\alpha+1}(X(u))dW(u)|_{-\alpha+1}$$

$$+ \int_s^t e^{\beta(t-u)}|\int_s^u T_{\alpha}(u-v)F^{1/2}_{-\alpha+1}(X(v))dW(v)|_{-\alpha+1} du$$

and the spatial regularity of $\int_0^{\cdot} T_{-\alpha}(\cdot-s)F^{1/2}_{-\alpha+1}(X(s))dW(s)$, as proved in Kotelenez [27] (generalizing a result of Dawson [10] - cf. also Kotelenez [25]). That the spatial regularity in (3.8) is maximal follows from the Gaussianity of $\int_0^t U_{-\alpha}(t,s)F^{1/2}_{-\alpha+1}(X(s))dW(s)$ - cf. for details Kotelenez [25].

(iii) The Markov property follows from Arnold, Curtain and Kotelenez [2], (3.9) follows from Curtain [6].

Remark 3.1

I. The final result can be expressed by

$$X_N = X + \frac{1}{\sqrt{\sqrt{N}}} Y + o\left(\frac{1}{\sqrt{\sqrt{N}}}\right) , \tag{3.10}$$

where X_N is the local stochastic, i.e., mezoscopic description, assuming $X_N(0)$ being near to homogeneity , X is the deterministic homogeneous state solution of (2.1), and Y is the mild solution of (3.7) which is a generalized Gauss-Markov process if Y_0 is Gaussian, and $o(\frac{1}{\sqrt{\sqrt{N}}})$ is the error term.

II. Let us now assume that we do not start in (2.1) with a constant but with some other positive bounded and possibly smooth function $X_0(q)$.

Then the difference $X_N(t) - X(t)$ satisfies

(3.11) $$X_N(t) - X(t) = F_N(t) + G_N(t),$$

where $F_N(t)$ is the r.h.s. in (2.13) and $G_N(t) = \int_o^t U_N(t,s)(D\Delta_N - D\Delta)X(s)ds$. Of course, (3.11) will also tend to zero under the assumptions of Theorem 3.1. However, in view of (2.18) we must normalize (3.11) by multiplying both sides by $(vN)^{1/2}$ (modulus a constant) in order to obtain a Gaussian correction term. On the other hand, the convergence of $(vN)^{1/2}G_N(t)$ to zero with $v = N^p$ and $p > 1$ does not hold(in general) in function norms and for $p < 2$ Arnold and Theodosopulu [3] have shown in the one-dimensional case that the variance of $Z_N(t)$ (the martingale part of $X_N(t)$ tends to ∞ in L_2-norm. This problem and related questions will be inves igated in a forthcoming paper.

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